$$\phi = (2/3A^2)[(1+2B-2^{1/2}A\eta)^{3/2}-1]-2B/A^2+2^{1/2}\eta/A \quad (4)$$

$$j_w = (-2^{1/2}/A)(1+2B)^{-1/2}+2^{1/2}/A \quad (5)$$

$$\eta_s = 2^{1/2}B/A \quad (6)$$

$$A = [\tfrac{2}{3}(1+2B)^{3/2}-\tfrac{2}{3}-2B]^{1/2} \quad (7)$$

This solution is algebraically more complicated than the one of Ref. 1 but does satisfy the boundary conditions. j was used as the basic parameter of the sheath solution in Ref. 1, and this is also true of the above solution except that it is more convenient to introduce the auxiliary parameter B, which is a function only of j as shown by Eqs. (5) and (7). In the limit of ζ large which corresponds to A and n_s large this solution and that of Ref. 1 give the same results. This conclusion is not too surprising since in the limit of ζ large the solution for the equivalent one-dimensional situation of Ref. 1 does satisfy the boundary condition that $\varepsilon_s = 0$.

In Ref. 3, solutions were obtained not only for the limiting case of $\omega = \infty$ but for finite values of ω and for both flat plates and conical probes. The results so obtained for flat plates and conical probes are shown in Figs. 1 and 2.

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[1] Johnson, R. A. and de Boer, P. C. T., "Theory of Ion Boundary Layers," *AIAA Journal*, Vol. 10, No. 5, May 1972, p. 664.

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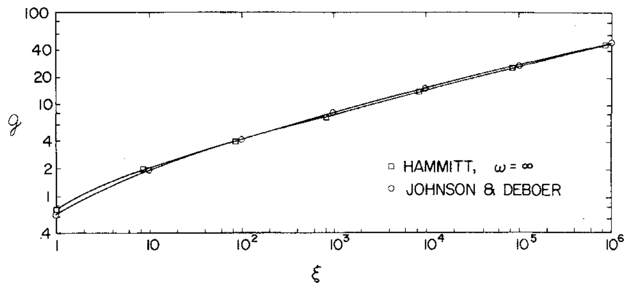

Fig. 1 Nondimensionalized current $\mathscr{J}$ as a function of ξ for flat plate probe. Squares are results of Ref. 2 for $\omega = \infty$; circles are results of Ref. 1.

Reply by Authors to A. G. Hammitt

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FIRST of all, we want to draw attention to the final two sentences of the next to last paragraph of the preceding comment by Hammitt. There, it is acknowledged that the objections raised in the comment do not apply to the downstream region (large ξ). In fact, the principal results of Refs. 1 and 2 for the flat plate are identical, and the subject of discussion is limited to "leading-edge corrections." Hammitt's implied contention that his leading-edge result constitutes an improvement over ours must be rejected as incorrect. This follows from the arguments given below, as well as from comparing the results of Refs. 1 and 2 directly (see Fig. 1). This comparison shows that the two results are in very close agreement.

Application to the leading-edge region (small and intermediate ξ) of "integral methods" such as used in Refs. 1 and 2 is not straightforward. The results obtained for this region must be regarded as empirical in nature. The difficulties go beyond the choice of a suitable approximate profile. Basically, the sheath cannot be regarded as thin near the leading edge, and a solution of the full two-dimensional equations is required. Hammitt treated this problem by "patching" his downstream solution to a solution based on assuming cylindrical symmetry around the origin. Clearly, the actual solution near the leading edge is not cylindrical, and this procedure provides no more than an estimation. In our opinion, use of the results of either Ref. 1 or 2 near the leading edge is justified mainly because of the good agreement with Dukowicz's numerical work[3] (see also Sec. II and Fig. 2 of Ref. 1).

As discussed in Ref. 5, our reasons for setting $\varepsilon_{\xi s} = -\eta_{s\xi}+j_\xi$ in Eq. (9) of Ref. 1 were as follows. Applying the equation $d\varepsilon = \varepsilon_\xi d\xi + \varepsilon_\eta d\eta = \varepsilon_\xi d\xi + n\, d\eta$ to the differential change of ε along the edge of the sheath, it is found that $\varepsilon_{\xi s} = -n_s\eta_{s\xi}+\varepsilon_{s\xi}$. Application of the boundary conditions, which already was made in setting $(n\varepsilon)_s = 0$ and $d\varepsilon_s = 0$, would yield $\varepsilon_{\xi s} = -\eta_{s\xi}$. On the other hand, use of the approximate profile of Ref. 1 gives $\varepsilon_{\xi s} = -\eta_{s\xi}+j_\xi$. For large ξ, the term j_ξ is negligibly small compared with $\eta_{s\xi}$, and the two results are equivalent. It follows that inclusion of the term j_ξ is of no consequence to the main result. The inclusion of j_ξ, which is empirical in nature, allows application of the initial condition $\eta_s = 0$ at $\xi = 0$. This obviates any need for "patching" the solution to some other result. As mentioned above, the results for $\mathscr{J}$ obtained this way agree well with the numerical work of Dukowicz.[3] When j_ξ is not retained in the expression for $\varepsilon_{\xi s}$, the solution for the current density j at small ξ is $j = C\exp(-\xi)$, where the constant C is undetermined. The "alternative choice" suggested by Hammitt, which consists of using our approximate profile also for $(n\varepsilon)_s$, is not a possible choice at all. It simply recovers the one-dimensional result, and its character is quite different from the approximation we used.

In order to establish the relation between Hammitt's work[2] and ours, we wish to point out that our theory already was available in printed form in January 1968 (Ref. 4) and also is contained in Ref. 5 which appeared in September 1969. It can easily be shown that the basic "integral" equation (34) of Ref. 2 is identical to the basic integral equation (9) of Ref. 1. While we do not disagree with Hammitt's argument that it is desirable to use an approximate profile that satisfies all of the boundary conditions, we note that Hammitt's profile does not lead to any essential improvement of the results. Apart from the comparison shown in Fig. 1, this follows by expanding the results in inverse powers of ξ, which can be compared with the exact result.[6] Both "integral methods" yield the lowest order term for $\mathscr{J}$ correctly, but the next higher order term incorrectly. The exact results[6] also show that neither the profile used by Hammitt[2] nor that used by us[1] is valid at small and intermediate ξ. As mentioned in the preceding comment, the profile used by Hammitt has the disadvantage of being more complicated algebraically than the one used by us.

In conclusion, the procedure we used in Ref. 1 to find leading-edge corrections to the main result is simpler and no less accurate than that of Ref. 2.

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Comment on "Lateral Vibration and Stability Relationship of Elastically Restrained Circular Plates"

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RAO and Amba-Rao[1] have presented the results of Rayleigh-Ritz calculations for the vibration frequencies of radially-compressed and elastically restrained circular plates and have shown that a plot of the buckling load vs a frequency parameter, λ, which they describe as the "natural frequency," is nonlinear. In fact, their λ is proportional to the square root of the natural frequency according to the definition given in their nomenclature, as may also be readily verified from the known solutions for the frequencies of circular plates without radial loading.[2] Moreover, there is no reason to expect the relationship between either the natural frequency or the square root of the natural frequency and the radial load to be linear, since the usual linear relationship is between the square of the frequency and the load.

The well-known relationship between compressive load and frequency squared which applies for the case of a column (and also for the critical speed of a compressed shaft) follows from Southwell's theorem[3] in the form of a lower bound

$$\omega^2/\omega_0^{\ 2} + P/P_0 \geqq 1$$

where P is the column load, P_0 is the column buckling load, ω is the lowest natural frequency and ω_0 is the lowest natural frequency in the absence of load P. The proof of this inequality is straightforward and is based on Rayleigh's principle.

For pin-ended columns, the deflection curves in vibration and in buckling are identical, and the equal sign applies. However, in a great many other cases, this lower-bound inequality is also very nearly an equality. Southwell's theorem also applies to the case of additional loadings which may in themselves cause instability. Care is needed in choosing the frequency and loading parameters in the governing differential equations or energy functionals so that Southwell's theorem will apply. In the case of the radially-compressed circular plate, it is clear from the energy functionals[2,4] that the linear inequality is between frequency squared and compressive load.

It turns out in this case, as in many others, that the Southwell inequality is extremely close to an equality so that a straight line is an excellent approximation to the curve of frequency-squared vs compressive load. (It is also worth noting that the inequality also applies to tensile load if the sign of P is reversed.) The near equality of the Southwell inequality may be readily demonstrated by calculating for the clamped plate the function ϕ defined by

$$\phi = R/R^* + (\lambda/\lambda^*)^4 \tag{1}$$

where the notation is that of Ref. 1, in which R is the compressive load, R^* is the buckling load, λ is the frequency parameter referred to above, and λ^* is the value of the frequency parameter in the absence of compressive load. This gives, using the data in Ref. 1, for R and λ

Table 1 Frequency-load relationship

R/R^*	$(\lambda/\lambda^*)^4$	ϕ
0	1.0	1.0000
0.1361	0.8690	1.0051
0.2721	0.7313	1.0034
0.4082	0.6030	1.0102
0.5442	0.4551	0.9993
0.6827	0.3258	1.0085
0.8163	0.1807	0.9970
0.9524	0.0404	0.9928
1.0	0	1.0000

More decimal places have been retained here than are warranted by the accuracy of the data in Ref. 1, so that the deviations of ϕ from unity are undoubtedly of the order of the round-off errors involved. Thus, to a high degree of approximation

$$R/R^* + (\lambda/\lambda^*)^4 = R/R^* + \omega^2/\omega_0^{\ 2} \approx 1.0 \tag{2}$$

and Southwell's theorem in this case may be regarded as an equality to an accuracy far greater than that required for all practical engineering purposes.

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Index categories: Structural Stability Analysis; Structural Dynamic Analysis.

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Reply by Authors to A. H. Flax

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WE thank Mr. Flax for his interesting comments. The present work concerning the vibration and stability relationship of structural elements is a part of a continuing program. Regretfully,

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Index category: Structural Stability Analysis.

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